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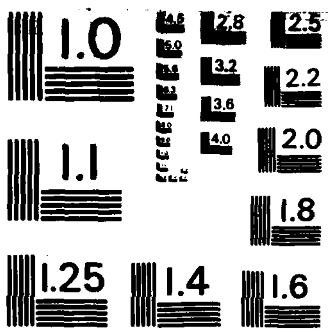

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COMMENT ON "CORRECTED DIFFUSION APPROXIMATIONS IN CERTAIN RANDOM WALK PROBLEMS"

by

Michael L. Hogan
Columbia University

TECHNICAL REPORT NO. 26
MAY 1984

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Abstract

Correction terms for the diffusion approximation to the maximum and ruin probabilities for a random walk with small negative drift, obtained by Siegmund [1979] in the exponential family case, are extended by different methods to some nonexponential family cases.

Key Words: Diffusion Approximation, Heavy Traffic, Random Walk, Gambler's Ruin.

an earlier work,

This paper is concerned with extensions to the nonexponential family case of two problems considered in ~~Siegmund [1979]~~. The first problem is to find the expected value of the maximum of a random walk with small, negative drift, and the second is to find the distribution of the same quantity. Siegmund's result in the first case is the following (Theorem 1 of [6]): Consider an exponential family P_θ, $\theta \in$ a neighborhood of 0, so that under P_θ, $x_1, x_2, \ldots$, are independent with density $e^{\theta x - \psi(\theta)}$ relative to a nonarithmetic distribution F. Assume that the problem is normalized so that $E_0(x_1) = \psi'(0) = 0$, $\mathrm{Var}_0(x_1) = \psi''(0) = 1$. Let $S_n = \sum^n x_i$, $\tau_b = \inf\{n: S_n > b\}$, $\tau_+ = \tau_0$, and $M = \sup_n\{S_n\}$, $M < \infty$ P_θ-a.e. if $\theta < 0$. Then, Siegmund shows that as $\theta \uparrow 0$,

$$E_\theta M = \frac{1}{\Delta} - \frac{E_0 S_{\tau_+}^2}{2E_0 S_{\tau_+}} + 0(\Delta),$$

where $\Delta = \theta_1 - \theta$, and $\theta_1 > 0$ is such that $\psi(\theta_1) = \psi(\theta)$. In fact, he can calculate the $0(\Delta)$ term, a feature that will not carry over to the nonexponential family case. In the second case the distribution of the maximum is given by considering such probabilities as $P_\theta\{\tau_b < \infty\} = P_\theta\{M > b\}$. The appropriate normalization in the exponential family case is to take $b = \frac{2\xi}{\Delta}$, in which case Siegmund showed that as $\theta \uparrow 0$

$$P_\theta\{\tau_{2\xi/\Delta} < \infty\} = e^{-2\xi}\left[1 - \Delta \frac{ES_{\tau_+}^2}{2ES_{\tau_+}} + o(\Delta)\right].$$

The first of the two problems has recently been considered by Klass [1983], who considers a translation family and computes the expected value of the maximum when the drift is small and negative up to terms that are $o(1)$ as the drift approaches 0, under the condition that the underlying random walk has third moments, which is certainly a minimal condition. The problem with Klass' correction term is that it does not resemble Siegmund's correction term. Even in the case of the normal distribution, to which both theorems apply, the equality of the two correction terms is not apparent.

Here, the random walks will also be assumed to belong to a translation family, i.e., $P_\theta\{X_1 \in A\} = P_0\{X_1 - \theta \in A\}$, where $E_0X_1 = 0$, $E_0X_1^2 = 1$, and $E_0|X_1^3| < \infty$. This is not necessary, but is the easiest framework, next to exponential families that incorporates the appropriate continuity in distribution. The maximum will again be denoted by $M = \sup\{S_i;\ i \geq 0\}$, which is almost surely finite provided $\theta < 0$. It will be shown that

$$E_\theta(M) = \frac{-1}{2\theta} + \frac{E_0S^2{}_{\tau_-}}{2E_0S_{\tau_-}} + o(1),$$

which is the result of Siegmund in the form he gave it, modulo a different parametri- zation. He used the parameter $\Delta = \theta_1 - \theta$, where $\theta < 0$ is the canonical parameter of the exponential family, and θ_1 is such that $\psi(\theta_1) = \psi(\theta)$. It is not hard to show that with $\psi(0) = \psi'(0) = 0$, and $\psi''(0) = 1$, $\frac{1}{\Delta} = -\frac{1}{2E_\theta X_1} + \frac{\gamma}{3} + o(1)$, $\gamma = E_0X_1^3$, and from the Wiener-Hopf factorization

$$\frac{E_0S_{\tau_+}^2}{2E_0S_{\tau_+}} + \frac{E_0S_{\tau_-}^2}{2E_0S_{\tau_-}} = \frac{\gamma}{3}.$$

Using these two relations, it is easy to establish the equivalence of Theorems 1 and 2 with the corresponding results of Siegmund.

The distribution of the maximum is determined by the quantities $P_\theta\{\tau_b < \infty\}$. To get a diffusion limit one assumes that $0 > \theta \to 0$, and $\theta b \to -\xi$, and the relation $b = m^{1/2}$, $\theta = \frac{-\xi}{m^{1/2}}$, $\xi > 0$, m a large integer has been chosen. This seems a bit peculiar here, but it is the usual way of normalizing a random walk to get a diffusion limit, and, in fact, the first term in the distribution is essentially given by the invariance principle. The result in this case is given by Theorem 2, and it basically depends on Lemma 1, but some technical conditions beyond those required by Lemma 1 have been imposed to facilitate a Fourier inversion. These conditions are, doubtless, not minimal.

In addition to the notation introduced above, for $\theta \leq 0$, let

$$\begin{aligned}
\tau_- &= \inf\{n : S_n \leq 0\}, \\
X_-(\lambda) &= E_\theta\{e^{i\lambda S_{\tau_-}}\}, \\
X_+(\lambda) &= E_\theta\{e^{i\lambda S_{\tau_+}}; \tau_+ < \infty\} \\
\phi(\lambda) &= E_\theta e^{i\lambda X_1}.
\end{aligned}$$

Of course, some of these quantities depend on θ, but that dependence will be suppressed.

Lemma 1.

$$E_\theta(S_{\tau_+}; \tau_+ < \infty)E_\theta(S_{\tau_-}) = -\frac{1}{2} + \frac{\theta E_0(S_{\tau_-}^2)}{2E_0(S_{\tau_-})} + o(\theta)$$

as $0 > \theta \to 0$, provided $E_0|X_1|^3 < \infty$.

Proof. From the Wiener-Hopf factorization (cf. Feller [1971], p. 605)

$$1 - \phi(\lambda) = (1 - X_-(\lambda))(1 - X_+(\lambda)).$$

In particular

$$\begin{aligned}
\phi'(0) &= X_-'(0)(1 - X_+(0)) \\
&= X_-'(0)P_\theta\{\tau_+ = \infty\}.
\end{aligned}$$

Consequently,

$$1 - \phi(\lambda) - (1 - X_-(\lambda))P_\theta\{\tau_+ = \infty\} = (1 - X_-(\lambda))(X_+(0) - X_+(\lambda)),$$

and

$$\begin{aligned}
&1 - \phi(\lambda) + \lambda X_-'(0)P_\theta\{\tau_+ = \infty\} - (1 - X_-(\lambda) + \lambda X_-'(0))P_\theta\{\tau_+ = \infty\} \\
&= (1 - X_-(\lambda))(X_+(0) - X_+(\lambda)).
\end{aligned}$$

The condition $E_\theta|X_1|^3 < \infty$ guarantees that $E_\theta(S^2_{\tau_-}) < \infty$, $E_\theta(S_{\tau_+};\ \tau_+ < \infty) < \infty$, so dividing by λ^2, and letting $\lambda \to 0$ produces

$$-\frac{\phi''(0)}{2} + \frac{\chi''_-(0)}{2} P_\theta\{\tau_+ = \infty\} = -E_\theta(S_{\tau_-})E_\theta(S_{\tau_+};\ \tau_+ < \infty)$$

or, using

$$P_\theta\{\tau_+ = \infty\} = \frac{1}{E_\theta\{\tau_-\}} = \frac{\theta}{E_\theta(S_{\tau_-})}$$

(Woodroofe [1982], Sec.2.3)

$$\begin{aligned} -E_\theta(S_{\tau_-})E_\theta(S_{\tau_+};\ \tau_+ < \infty) &= \frac{1}{2}\left(E_\theta(X_1^2) - \frac{\theta E_\theta S^2_{\tau_-}}{E_\theta S_{\tau_-}}\right) \\ &= \frac{1}{2}\left(E_0(X_1^2) - \frac{\theta E_\theta(S^2_{\tau_-})}{E_\theta(S_{\tau_-})}\right) + o(\theta^2). \end{aligned}$$

Now, $\dfrac{E_\theta(S^2_{\tau_-})}{E_\theta(S_{\tau_-})}$ is a continuous function of θ (see proof of Theorem 2, Chapter 5, Hogan [1984]), in spite of the fact that numerator and denominator separately need not be. This observation establishes the lemma.

Theorem 1. Suppose $E_0|X_1^3| < \infty$. Then as $0 > \theta \to 0$

$$E_\theta M = -\frac{1}{2\theta} + \frac{E_0 S^2_{\tau_-}}{2E_0 S_{\tau_-}} + o(1).$$

Proof. It is easy to see that M can be represented as a randomly stopped random walk, $M \stackrel{\mathcal{L}}{=} Z_N$, where, under P_θ, $Z_1 \stackrel{\mathcal{L}}{=} S_{\tau_+} \mid \tau_+ < \infty$, and N is a geometric random variable with success probability $P_\theta\{\tau_+ < \infty\}$ completely independent of the random walk. Hence

$$\begin{aligned}
E_\theta M &= E_\theta N E_\theta(S_{\tau_+} \mid \tau_+ < \infty) \\
&= \frac{E_\theta(S_{\tau_+};\ \tau_+ < \infty)}{P_\theta\{\tau_+ = \infty\}} \\
&= E_\theta(S_{\tau_+};\ \tau_+ < \infty) E_\theta\{\tau_-\} \\
&= E_\theta(S_{\tau_+};\ \tau_+ < \infty) E_\theta(S_{\tau_-}) \cdot \frac{1}{\theta} \\
&= -\frac{1}{\theta}\left\{\frac{1}{2} - \theta\frac{ES_{\tau_-}^2}{2ES_{\tau_-}} + o(\theta)\right\}
\end{aligned}$$

by Lemma 1. ∎

The increasing ladder times of a random walk S_i are the times

$$\tau^{(0)} = 0,$$

$$\tau^{(1)} = \inf\{n :\ S_n > 0\},$$

$$\tau^{(n+1)} = \begin{cases} \inf\{n > \tau^{(n)} :\ S_n > S_{\tau^{(n)}},\}, & \tau^{(n)} < \infty \\ \infty, & \tau^{(n)} = \infty. \end{cases}$$

The increasing ladder process is the process

$$Z_0 = 0$$

$$Z_i = S_{\tau^{(i)}} 1_{(\tau^{(i)} < \infty)}, \qquad i > 0.$$

Lemma 2. Let S_i be a random walk with $ES_1 < 0$. Then

$$P\{S_i \le a\ \forall i\} = F(a)\, P\{S_i \le 0\ \forall i\},$$

where $F(a) = \sum_{i=0}^{\infty} P\{Z_i \le a,\ \tau^{(i)} < \infty\}$ and Z_i is the increasing ladder process of S_i.

Proof. This is Lemma 8 of Spitzer [1957] when the distribution of X_i is absolutely continuous and is equivalent to 12.2.7 of Feller [1971] in the general case.

Theorem 2. Suppose X_i are i.i.d. random variables satisfying $E_0 X_i = 0$, $E_0 X_i^2 = 1, E_0|X_1|^5 < \infty$, and P_θ are such that $P_\theta\{X_i \in A\} = P_0\{X_i - \theta \in A\}$. Suppose further that $\exists \theta_0 < 0$ such that $\forall\, \theta \in (\theta_0, 0]$, $S_{\tau_+} 1_{\{\tau_+ < \infty\}}$ has an absolutely continuous distribution under P_θ and satisfies

Condition 0: The densities $\frac{d}{dx} P_\theta\{S_{\tau_+} < x\}$ are a bounded subset of $L^\lambda(\Re)$ for some $\lambda > 1$. Then

$$P_{-\theta}\{\tau_{\xi/\theta} < \infty\} = e^{-2\xi}\left(1 + \frac{4\xi\gamma\theta}{3} - \frac{\theta E_0 S_{\tau_+}^2}{E_0 S_{\tau_+}}\right) + o(\theta)$$

as $\theta \uparrow 0$.

Proof. Many quantities will implicitly depend on the parameter. Let

$$p = P_{-\theta}\{\tau_+ < \infty\}$$
$$\mu_i = E_{-\theta}(S_{\tau_+}^i | \tau_+ < \infty)$$
$$f(t) = E_{-\theta}(e^{iS_{\tau_+} t} | \tau_+ < \infty).$$

Consider first the quantity $\sum_{n=0}^{\infty} P_{-\theta}\{Z_n < \frac{\xi}{\theta},\ \tau^{(n)} < \infty\}$, where Z_n is the increasing ladder process for S_i, and the $\tau_{(n)}$ are the increasing ladder times. As in the Fourier-analytic proof of the Renewal Theorem (see e.g. Breiman [1968]) it is easy to show that

$$\begin{aligned}
\sum_{n=0}^{\infty} P\{Z_n < \frac{\xi}{\theta},\ \tau^{(n)} < \infty\} &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \xi t/\theta}{t} \frac{dt}{1 - p\, f(t)} \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \xi t/\theta}{t} Re\left(\frac{1}{1 - p\, f(t)}\right) dt \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \xi t/\theta}{t} \frac{(1 - p\, f_1(t))dt}{(1 - p\, f_1(t))^2 + (p f_2(t))^2}
\end{aligned}$$

where $f(t) = f_1(t) + i f_2(t)$, f_1, f_2 real. Making the change of variable $\frac{t}{\theta} = z$ and multiplying and dividing by $1-p$ brings this into the form

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \xi z}{z(1-p)} \frac{(1-p\, f_1(\theta z))(1-p)}{(1-p\, f_1(\theta z))^2 + (f_2(\theta z))^2}\, dz,$$

and so by Lemma 2

$$\begin{aligned}
P_\theta\{\tau_a = \infty\} &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \xi z}{z} \frac{dz}{\frac{1-p\, f_1(\theta z)}{1-p} + \frac{(p f_2(\theta z))^2}{(1-p)(1-p\, f_1(\theta z))}} \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \xi z}{z}\left(\frac{1}{1+\nu_p^2 z^2}\right) dz \\
&+ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \xi z}{z}\left(\frac{1}{\frac{1-p\, f_1(\theta z)}{1-p} + \frac{(p(f_2(\theta z))^2}{(1-p)(1-p\, f_1(\theta z))}} - \frac{1}{1-\nu_p^2 z^2}\right) dz
\end{aligned} \tag{1}$$

where $\nu_p = E_{-\theta}(S_{\tau_-})E_{-\theta}(S_{\tau_+};\ \tau_+ < \infty)$. By Lemma 1, ν_p is known up to terms which are $o(1-p)$, and therefore the first integral is known up to terms of the same order. It is necessary to analyze $\frac{1}{1-p} \times$ second integral. For this, it is convenient to break the integral up into $\int_{\xi\theta^\alpha}^{\xi\theta^\alpha} + \int_{|z|>\xi\theta^\alpha}$, where ξ, $-\alpha > 0$ will be specified later. The conditions of the theorem guarantee that f_1 and f_2 have expansions of the form

$$\begin{aligned}
f_1 &= 1 - \frac{\mu_2 z^2}{2} + O(z^4) \\
f_2^2 &= \mu_1^2 z^2 + O(z^4)
\end{aligned}$$

where $O(z^k)$ holds uniformly. Using the fact that $\frac{\theta}{1-p}$ is bounded above and away from 0 it is easy to see that

$$\begin{aligned}
\frac{1-p f_1(\theta z)}{1-p} &= 1 - p\frac{\mu_2}{2}\frac{\theta}{1-p}\theta z^2 + O(\theta^3 z^4) \\
\frac{f_2^2(\theta z)}{(1-p)(1-p\, f_1(\theta z))} &= \frac{\mu_1^2 z^2 (\frac{\mu}{1-p})^2 + O(\theta^2 z^4)}{1 - p\frac{\mu_2}{2}\frac{\mu}{1-p}\theta z^2 + O(\theta^3 z^4)}
\end{aligned}$$

and

$$\frac{p^2 f_2^2(\theta z)}{(1-p)(1-p\, f_1(\theta z))} = \frac{\nu_p^2 z^2 + o(\theta^2 z^4)}{1 - p\frac{\mu_2}{2}\frac{\theta}{1-p}\theta z^2 + O(\theta^3 z^4)}$$
$$= (\nu_p^2 z^2 + O(\theta^2 z^4))\Big(1 + p\frac{\mu_2}{2}\frac{\theta}{1-p}\theta z^2 + O(\theta^2 z^4)\Big)$$
$$= \nu_p^2 z^2 + p\frac{\mu_2}{2}\frac{\theta}{1-p}\theta z^2 \cdot \nu_p^2 z^2 + O(\theta^2 z^6).$$

Consequently

$$R\frac{1-p}{1-p\, f(\theta z)} = \frac{1}{1 + \nu_p^2 z^2 - p\frac{\mu_2}{2}\frac{\theta}{1-p}\theta z^2(1-\nu_p^2 z^2) + O(\theta^2 z^6)}$$

and

$$\frac{1}{\theta}\left(Re\frac{1-p}{1-p\, f(\theta z)} - \frac{1}{1+\nu_p^2 z^2}\right) = \frac{p\frac{\mu_2}{2}\frac{\theta}{1-p} z^2(1-\nu_p^2 z^2) + O(\theta z^5)}{(1+\nu_p^2 z^2)(1 + \nu_p^2 z^2 + O(\theta^2 z^6))}.$$

Now, if $\alpha > -\frac{1}{2}$ the $O(\theta^2 z^6)$ term in the denominator is $o(z^2)$ for $|z| < \epsilon\theta^{\alpha}$, and consequently the denominator is $\geq$ const.$(1+z^4)$. Therefore, the second term in the numerator contributes to the integral as

$$\int_{|z|<\epsilon\theta^{\alpha}} \frac{\sin \xi z}{z}\frac{(\theta z^2)\cdot z^3}{1+z^4}\, dz,$$

and $\theta z^2 \to 0$ as $\theta \to 0$, $|z| < \epsilon\theta^{\alpha}$. Hence, by dominated convergence, this contribution $\to 0$.

It is also clear that

$$\frac{1}{(1+\nu_p^2 z^2)} - \frac{1}{1+\nu_p^2 z^2 + O(\theta^2 z^6)} = O(\theta^2 z^2) = \theta^{\delta}\, O(z^{-\delta})$$

for some $\delta > 0$, and so

$$\int_{|z|<\epsilon\theta^\alpha} \frac{\sin \xi z}{z} \cdot \frac{p\frac{\mu_2}{2}\frac{\theta}{1-p} z^2(1-\nu_p^2 z^2)}{(1+\nu_p^2 z^2)(1+\nu_p^2 z^2 + O(\theta^2 z^6))} dz$$

$$= \int_{|z|<\epsilon\theta^\alpha} \frac{\sin \xi z}{z} \frac{p\frac{\mu_2}{2}\frac{\theta}{1-p} z^2(1-\nu_p^2 z^2)}{(1+\nu_p^2 z^2)^2} dz$$

$$+ \theta^\delta \int_{|z|<\epsilon\theta^\alpha} \frac{\sin \xi z}{z} \frac{p\frac{\mu_2}{2}\frac{\theta}{1-p} z^2(1-\nu_p^2 z^2) z^{-\delta}}{(1+\nu_p^2 z^2)(1+\nu_p^2 z^2 + O(\theta^2 z^6))} dz$$

and the second term is easily seen to $\to 0$. Consequently, using $\mu_2 \frac{\theta}{1-p} \to \beta = \frac{E\, S_{\tau_+}^2}{2E\, S_{\tau_+}}$, $\nu_p^2 \to \frac{1}{4}$,

$$\frac{1}{\theta} \int_{|z|<\epsilon\theta^\alpha} \frac{\sin \xi z}{z} \cdot \left(\frac{1-p}{1-p\, f(\theta z)} - \frac{1}{1-\nu_p^2 z^2} \right) dz = \frac{\beta}{2} \int_{-\infty}^{\infty} \frac{\sin \xi z}{z} \frac{z(1-(\frac{z}{2})^2)}{(1+(\frac{z}{2})^2)^2} dz$$

where this last integral exists, and is taken as a principal value. Evaluating shows this term to be

$$\beta(4\xi - 2)e^{-2\xi}. \tag{2}$$

Next ϵ will be chosen to satisfy $\cos \xi\epsilon\theta^\alpha = 0$. Clearly, since $\alpha < 0$ such that ϵ can be chosen arbitrarily small (or large). Then integrating by parts shows that

$$\left|\frac{1}{\theta} \int_{|z|>\epsilon\theta^\alpha} \frac{\sin \xi z}{z} \cdot \frac{dz}{1+\nu_p^2 z^2}\right| = 2\left|\frac{1}{\theta} \int_{\epsilon\theta^\alpha}^{\infty} \frac{\cos \xi z(1+3z^2\nu_p^2)}{z^2(1+\nu_p^2 z^2)^2} dz\right|$$

$$\leq \text{const.}\ \frac{1}{\theta} \cdot (\epsilon\theta^{-\alpha})^3$$

$\to 0$ provided $\alpha < -\frac{1}{3}$.

Finally, consider

$$\left|\frac{1}{\theta} \int_{|z|>\theta^\alpha\epsilon} \frac{\sin \xi z}{z} \frac{1-p}{1-p\, f(\theta z)} dz\right| = |E(S_{\tau_-}) \int_{|z|>\epsilon\theta^\alpha} \frac{\sin \xi z}{z} \cdot \frac{1}{1-p\, f(\theta z)} dz|.$$

The change of variable $y = \theta z$ brings the integral to the form

$$\begin{aligned}\left|\int_{|y|>\epsilon\theta^{1+\alpha}} \frac{\sin\frac{\xi z}{\theta}}{z}\cdot\frac{dz}{1-p\,f(\theta z)}\right| &\le \left|\int_{\epsilon\theta^{1+\alpha}}^{\infty}\frac{\sin\frac{\xi z}{\theta}}{z}\,dz\right| \\ &\quad + \left|\int_{\epsilon\theta^{1+\alpha}}^{\infty}\frac{\sin\xi z/\theta}{z}\cdot\frac{pf(z)}{1-pf(z)}\right| \\ &= I + II.\end{aligned}$$

With the choice of ϵ as above

$$\begin{aligned}\left|\int_{\epsilon\theta^{1+\alpha}}^{\infty}\frac{\sin\frac{\xi z}{\theta}}{z}\right| &= \frac{\theta}{\xi}\left|\int_{\epsilon\theta^{1+\alpha}}^{\infty}\frac{\cos\xi z/\theta}{z^2}\,dz\right| \\ &\le \text{const. } \theta\cdot\theta^{-\alpha-1} \to 0,\end{aligned}$$

$$II = \int_{\epsilon\theta^{1+\alpha}}^{1/2\,\mu_1} + \int_{\frac{1}{2\mu_1}}^{\infty}.$$

Condition O has two consequences. First, because under P_θ, $S_{\tau_+}1_{(\tau_+<\infty)}$ has a continuous distribution it is continuous in distribution as a function of θ. This is easy to see simply by considering sample paths. In particular, given a sequence

$$f_i(t) = E_{\theta_i}\{e^{it\,S_{\tau_+}};\ \tau_+ < \infty\} \quad \theta_i \in (\theta,\ 0],$$

it is possible to find a pointwise convergent subsequence. Second, by the Hausdorff-Young theorem (Katznelson [1976], p. 142) the functions $f(y)$ form a bounded subset in $L^{\frac{\lambda}{1-\lambda}}(\Re, dy)$. In particular, if $1 < \lambda_0 < \frac{\lambda}{1-\lambda}$, $|f|^{\lambda_0}$ are uniformly integrable. In as much as any subcollection has a pointwise convergent subsequence, it has an $L^{\lambda}(\Re, dy)$ convergent subsequence, and so the $f(y)$ form a compact subset of $L^{\lambda_0}(\Re,\ dy)$. The function

$$\frac{1}{y}1_{(y>\frac{1}{2\mu_1})} \in L^{\frac{\lambda_0}{1-\lambda_0}}(\Re,\ dy)$$

and so

$$\left\{\frac{f(y)}{y}1_{(y>\frac{1}{2\mu_1})}\right\}$$

is a compact subset of $L^1(\Re,\ dy)$ (reminder: f depends implicitly on θ). It follows directly from the Riemann-Lebesgue lemma that

$$\int_{\frac{1}{2\mu_1}}^{\infty}\frac{\sin\frac{\xi y}{\theta}}{y}\frac{f(y)}{1-p\,f(y)}dy \to 0.$$

For $\int_{\epsilon\theta^{1+\alpha}}^{\frac{1}{2}\mu_1}(same)dz$, note that

$$\frac{f(z)}{z(1-p\,f(z))}=\frac{1+i\mu_1 z}{z(1-p(1+i\mu_1 z))}+g(z),\ z\in[0,\frac{1}{2\mu_1}],$$

where $g(z)$ is uniformly bounded and continuous in θ. It follows as above that

$$\int_{\epsilon\theta^{1+\alpha}}^{1}\sin\left(\frac{\xi z}{\theta}\right)g(z)dz\to 0.$$

Also, it is clearly enough to consider

$$Re\int_{\epsilon\theta^{1+\alpha}}^{\frac{1}{2}\mu_1}\frac{\sin\frac{\xi z}{\theta}}{z}\cdot\frac{1+i\mu_1 z}{(1-p)-ip\mu_1 z}dz$$
$$=\int_{\epsilon\theta^{1+\alpha}}^{\frac{1}{2\mu_1}}\frac{\sin\frac{\xi z}{\theta}}{z}\cdot\frac{(1-p)+p\mu_1^2 z^2}{(1-p)^2+(p\mu_1 z)^2}$$
$$\frac{p^2(\mu_1 z)^2}{(1-p)^2+p^2\mu_1^2 z^2}=1-\frac{(1-p)^2}{(1-p)^2+p^2\mu_1^2 z^2}$$

and since

$$\int_{\epsilon\theta^{1+\alpha}}^{\frac{1}{2\mu_1}}\frac{\sin\left(\frac{\xi z}{\theta}\right)dz}{z}$$

has been treated it is enough to consider

$$(1-p)\int_{\epsilon\theta^{1+\alpha}}^{\frac{1}{2\mu_1}} \frac{\sin\frac{\xi z}{\theta}}{z}\cdot\frac{1}{(1-p)^2+(p\mu_1 z)^2}\,dz$$
$$=(1-p)\cdot\theta\left[\int_{\epsilon\theta^{1+\alpha}}^{\frac{1}{2\mu_1}}\cos\frac{\xi z}{\theta}\cdot\left\{\frac{1}{z^2((1-p)^2+(p\mu_1 z)^2)}+\frac{2p\mu_1 z}{z((1-p)^2+(p\mu_1 z)^2)^2)}\right\}dz\right].$$

Over the interval of integration $(1-p)$ is much smaller than z, so each term in the integrand is $< \frac{\text{const.}}{z^4}$, and the integral is bounded by

$$\text{const.}(1-p)\cdot\theta\cdot\int_{\epsilon\theta^{1+\alpha}}^{\frac{1}{2\mu_1}}\frac{dz}{z^4}\leq \text{ const. }\theta^2(\theta^{-3(1+\alpha)})$$

$\rightarrow 0$ provided $\alpha < -\frac{1}{3}$. Therefore, the various conditions placed on α are simultaneously satisfied for any $\alpha \in (-\frac{1}{2}, -\frac{1}{3})$. So, evaluting the integral in (1) and combining that result with (2) gives

$$P\{\tau_{\xi/\theta}=\infty\}=e^{-\frac{\xi}{\nu_p}}+\theta\beta(4\xi-2)e^{-2\xi}+o(\theta).$$

By Lemma 1

$$\frac{1}{\nu_p}=2+4\theta\frac{E(S_{\tau_-}^2)}{2E(S_{\tau_-})}+o(\theta)$$

so

$$P\{\tau_{\xi/\theta}=\infty\}=e^{-2\xi}\left(1-4\xi\theta\left(\frac{E(S_{\tau_-}^2)}{2E(S_{\tau_-})}+\beta\right)-2\theta\beta+o(\theta)\right).$$

The Wiener-Hopf factorization shows that

$$\frac{ES_{\tau_-}^2}{2E(S_{\tau_-})}+\beta=\frac{\gamma}{3},$$

where $\gamma = E_0(X_1^3)$. Hence

$$P\{\tau_{\xi/\theta} = \infty\} = e^{-2\xi}\left(1 - \frac{4\xi\gamma\theta}{3} - 2\theta\beta\right) + o(\theta),$$

which is the form given by Siegmund [1979], modulo a different choice of parameters, as explained earlier.

Here is one condition that guarantees that condition O is met. Other conditions along this line can be formulated.

Proposition 1. Suppose X_1 has a density under P_0 which is bounded, and decreasing on $[0, \infty)$. Then condition **0** is satisfied.

Proof. First note that the existence of a density for S_{τ_+} is trivial as any randomly stopped partial sum of an absolutely continuous random walk is absolutely continuous. It is certainly enough to show that the densities are uniformly bounded by a constant. These densities can be written down explicitly.

$$\begin{aligned} P\{S_{\tau_+} > x,\ \tau_+ < \infty\} &= \sum_{n=0}^{\infty} \int_{-\infty}^{0} P\{X_1 > x - y\}\, P\{S_n \in dy,\ \tau_+ > n\} \\ &= \int_{-\infty}^{0} P\{X_1 > x - y\} H(dy) \end{aligned}$$

where

$$\begin{aligned} H(A) &= \sum_{n=0}^{\infty} \{S_n \in A,\ \tau_+ > n\},\ A \subset (-\infty, 0] \\ &= \sum_{n=0}^{\infty} \{S_1 \le 0, \ldots, S_{n-1} \le 0,\ S_n \in A\} \\ &= \sum_{n=0}^{\infty} \{S_n \le S_1, \ldots, \ldots, S_n \le S_{n-1},\ S_n \in A\} \\ &= E\{\#\text{ of visits of weak decreasing ladder process to } A\}. \end{aligned}$$

Consequently

$$\frac{P\{S_{\tau_+} < x+h\} - P\{S_{\tau_+} < x\}}{h}$$
$$= \int_{-\infty}^{0} \frac{P\{X_1 < x+h-y\} - P\{X_1 < x-y\}}{h} H(dy).$$

The integrand is $\leq f(x-y)$ except possibly in a small neighborhood of the origin where it is still bounded and

$$\int_{-\infty}^{0} f(x-y)\, H(dy) \leq \sum_{n=0}^{\infty} f(x+n)\, H(-n-1,\, -n)$$
$$\leq \text{ const.} \sum f(x+n) < \infty,$$
$$\leq \text{ const.} \sum f(n) \leq \text{ const.}$$

therefore, by dominated convergence

$$\frac{d}{dx} P\{S_{\tau_+} < x\} = \int_{-\infty}^{0} f(x-y)\, H(dy) \leq \text{ const.}$$

H depends on the parameter p, but it is easy to see that the statements made about H hold uniformly.

The representation of the distribution of S_{τ_+} used in Proposition 1 can also be used to prove a conjecture found in Klass [1983], remark 2.5, which, in the notation used here is that if $E_0(S_1^2;\, S_1 > 0) < \infty$, $E_0 S_1 = 0$, then

$$\lim_{\theta \uparrow 0} E_\theta(S_{\tau_+};\, \tau_+ < \infty) = E_0(S_{\tau_+}).$$

Klass observes that pointwise convergence takes place. The following bound on the distributions imply uniform integrability.

$$P_\theta\{S_{\tau_+} > n,\, \tau_+ < \infty\} = \int_{-\infty}^{0} P_{-\theta}\{S_1 > n-y\}\, H(dx) \leq \text{ const.} \sum_{n}^{\infty} P_{-\theta}\{S_1 > n\}.$$

This stochastically bounds $S_{\tau_+} 1_{(\tau_+ < \infty)}$ by a fixed integrable random variable.

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BIBLIOGRAPHY

[1] Breiman, L. (1968), **Probability**, Addison-Wesley Publishing Co., Reading, MA.

[2] Feller, W. (1971), **An Introduction to Probability Theory and Its Applications**, John Wiley and Sons, New York.

[3] Hogan, M. H. (1984), **Some Problems in Boundary Crossing for Random Walks**, Ph.D. Dissertation, Department of Statistics, Stanford University.

[4] Katznelson, Y. (1976), **An Introduction to Harmonic Analysis**, Dover Publications, New York.

[5] Klass, M. (1983), *On the Maximum of a Random Walk with Small Negative Drift*, **Ann. Prob. 11**, pp. 491-505.

[6] Siegmund, D. (1979), *Corrected Diffusion Approximations in Certain Random Walk Problems*, **Adv. Appl. Prob. 11**, p. 701-719.

[7] Spitzer, F. (1957), *The Wiener-Hopf Equation Whose Kernel is a Probability Density*, **Duke J. Math. 27**, pp. 363-372.

[8] Woodroofe, M. (1982), **Nonlinear Renewal Theory in Sequential Analysis**, SIAM Publications, Philadelphia, PA.

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